Eur. Phys. J. C 28, 115–118 (2003)
Digital Object Identifier (DOI) 10.1140/epjc/s2003-01143-3

THE EUROPEAN
PHYSICAL JOURNAL C

Asymptotic conditions for the electromagnetic form factors of hadrons represented by the VMD model

C. Adamuščin[1], A.-Z. Dubničková[1], S. Dubnička[2], R. Pekárik[2], P. Weisenpacher[3]

[1] Department of Theoretical Physics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovak Republic
[2] Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovak Republic
[3] Institute of Informatics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 37 Bratislava, Slovak Republic

Received: 20 March 2002 / Revised version: 3 December 2002 /
Published online: 7 March 2003 –

Abstract. A system of linear homogeneous algebraic equations for the coupling constant ratios of vector mesons to hadrons is derived by imposing the assumed asymptotic behavior upon the VMD pole parameterization of an hadron electromagnetic form factor. A similar system of equations with a simpler structure of the coefficients, taken as even powers of the vector-meson masses, is derived by means of integral superconvergent sum rules for the imaginary part of the considered form factor using its appropriate δ-function approximation. Although both systems have been derived starting from different properties of the electromagnetic form factor and they each have their own appearances, it is shown explicitly that they are fully equivalent.

1 Introduction

Hadrons are complex systems with many internal degrees of freedom and their constituents are quarks and gluons interacting by the laws of QCD. As a result hadrons have a finite size, which in EM interactions is revealed as the EM structure of the hadrons, phenomenologically to be described by functions of one variable (the four-momentum transfer squared t of a virtual photon), called the EM form factors (FFs). The most simple and frequently used framework to parameterize the EM FFs is the zero-width vector-meson dominance (VMD) model [1,2] based on the effective Lagrangian of the quantum field theory, in which one assumes that the virtual photon (after having become a quark–antiquark pair) couples to the hadron as a stable vector meson. The EM FFs can then be expressed in terms of the vector-meson masses m_v, the coupling strengths between the virtual photon and the vector meson $g_{\gamma^* v} = (em_v^2)/f_v$ and between the vector meson and the considered hadron f_{vhh} and finally summing over all possible n vector mesons:

$$F_h(t) = \sum_{v=1}^{n} \frac{m_v^2}{m_v^2 - t}(f_{vhh}/f_v). \qquad (1)$$

The FF in this form is a pure real function in the whole physical region $-\infty < t < +\infty$ with the poles on the positive real axis. It does not actually have analytic properties and it does not obey the unitarity condition.

In fact the VMD model is unable to reproduce the existing experimental information on the EM FFs of hadrons properly, especially in the time-like region, where various vector-meson resonances are created in the e^+e^--annihilation processes into hadrons and only its unitarization, i.e., the inclusion of the continua contributions and the instability of vector-meson resonances, leads to a simultaneous description of the space-like and time-like data.

A further shortcoming of the VMD models is that they do not predict a correct behavior of the EM FFs at high $|t|$ values. The quark–dimensional scaling framework [3, 4] predicts that only the number of valence quarks n_q of the hadron determines the asymptotic behavior of the EM FFs:

$$F_h(t)_{|t|\to\infty} \sim t^{1-n_q}, \qquad (2)$$

and so it is determined just by the number of gluon propagators.

From (2) one can see immediately that for hadrons with $n_q > 2$ the asymptotic behavior of their EM FFs is different from the asymptotic behaviors of (1) and in the construction of the unitary and analytic models [5–8] of the EM structure of such hadrons, in which the VMD model is a starting point, it has to be properly adopted.

Nevertheless, starting from different properties of the EM FF of the hadron one can derive two apparently distinct systems of asymptotic conditions for the EM FF of a hadron represented by the VMD model. The first one is derived in the absolutely correct way by transforming the VMD pole representation (1) into a common denominator and putting the necessary number of coefficients from the highest powers of t in the numerator to zero. Such a system of linear homogeneous algebraic equations for the coupling constant ratios has, however, coefficients being rather complicated sums of products of vector-meson masses squared and one is not be able to find its solutions in the general case. The second system is obtained from integral superconvergent sum rules for the imaginary part of the EM FF of a hadron, derived on the basis of the true analytic properties of the EM FF of the hadron under consideration; these have nothing in common with the VMD pole parameterization of the considered EM FF and in our opinion the whole problem is not well founded. The advantage is that the obtained system of linear homogeneous algebraic equations for the coupling constant ratios has coefficients that are simply even powers of the corresponding vector-meson masses and in principle one can find its solutions even in the general case.

The dilemma described above is solved in this paper by proving generally that both systems of asymptotic conditions are fully equivalent.

This paper is organized as follows. In the next section we derive two systems of $(m-1)$ linear homogeneous algebraic equations starting from different properties of the electromagnetic FF of the hadron. To an explicit proof of their equivalence is devoted Sect. 3. In the last section we present our conclusions and a discussion.

2 Algebraic equations for the coupling constant ratios

Generally, let us assume that the FF in (1) is saturated by n different vector-meson pole terms and that the asymptotic behavior

$$F_{h_{|t|\to\infty}} \sim t^{-m}, \tag{3}$$

is required, where $m \leq n$.

Then transforming the VMD pole representation (1) into a common denominator, one obtains the FF in the form of a rational function with a polynomial of degree $(n-1)$ in the numerator, and putting in the latter the first $(m-1)$ coefficients from the highest powers of t to zero, one obtains the first system of linear homogeneous algebraic equations for the coupling constant ratios $a_j = (f_{jhh}/f_j)$:

$$\begin{aligned}
&\sum_{j=1}^{n} m_j^2 a_j = 0,\\
&\sum_{\substack{i=1\\ i\neq j}}^{n} m_i^2 \sum_{j=1}^{n} m_j^2 a_j = 0,\\
&\sum_{\substack{i_1,i_2=1\\ i_1<i_2,\, i_r\neq j}}^{n} m_{i_1}^2 m_{i_2}^2 \sum_{j=1}^{n} m_j^2 a_j = 0,\\
&\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3,\, i_r\neq j}}^{n} m_{i_1}^2 m_{i_2}^2 m_{i_3}^2 \sum_{j=1}^{n} m_j^2 a_j = 0,\\
&\cdots\cdots\cdots\\
&\sum_{\substack{i_1,i_2,\cdots i_{m-2}=1\\ i_1<i_2\cdots<i_{m-2},\, i_r\neq j}}^{n} m_{i_1}^2 m_{i_2}^2 \cdots m_{i_{m-2}}^2 \sum_{j=1}^{n} m_j^2 a_j = 0.
\end{aligned} \tag{4}$$

Here with increased m the coefficients become sums of more and more complicated products of squared vector-meson masses.

For a derivation of the second system we employ the assumed analytic properties of the EM FFs of the hadrons, consisting of an infinite number of branch points on the positive real axis, i.e. cuts, where the first one extends starting from the lowest branch point t_0 to $+\infty$. Then applying the Cauchy theorem to $F_h(t), tF_h(t), t^2F_h(t), \cdots, t^{m-2}F_h(t)$ with the closed integration path consisting of the circle C_R of the radius $R \to \infty$ and the path avoiding the cuts on the positive real axis, one gets $(m-1)$ superconvergent sum rules:

$$\begin{aligned}
&\frac{1}{\pi}\int_{t_0}^{\infty} \mathrm{Im}F_h(t)\mathrm{d}t = 0,\\
&\frac{1}{\pi}\int_{t_0}^{\infty} t\cdot\mathrm{Im}F_h(t)\mathrm{d}t = 0,\\
&\frac{1}{\pi}\int_{t_0}^{\infty} t^2\cdot\mathrm{Im}F_h(t)\mathrm{d}t = 0,\\
&\cdots\cdots\cdots\\
&\frac{1}{\pi}\int_{t_0}^{\infty} t^{m-2}\cdot\mathrm{Im}F_h(t)\mathrm{d}t = 0.
\end{aligned} \tag{5}$$

Now, approximating the FF's imaginary part by a δ-function in the form as follows:

$$\mathrm{Im}F(t) = \pi\sum_{i}^{n} a_i\delta(t-m_i^2)m_i^2 \tag{6}$$

and substituting it into (5), one obtains the second system of $(m-1)$ linear homogeneous algebraic equations for the coupling constant ratios $a_i = (f_{ihh}/f_i)$

$$\begin{aligned}
&\sum_{i=1}^{n} m_i^2 a_i = 0,\\
&\sum_{i=1}^{n} m_i^4 a_i = 0,\\
&\sum_{i=1}^{n} m_i^6 a_i = 0,\\
&\cdots\cdots\cdots\\
&\sum_{i=1}^{n} m_i^{2(m-2)} a_i = 0,\\
&\sum_{i=1}^{n} m_i^{2(m-1)} a_i = 0,
\end{aligned} \tag{7}$$

however, with coefficients that are simply even powers of the vector-meson masses.

In the next section we demonstrate explicitly that both systems of algebraic equations, (4) and (7), are equivalent.

3 Equivalence of systems of algebraic equations for the coupling constants ratios

In this section we show step by step that the systems of linear algebraic equations (7) and (4), despite the fact that they have been derived starting from different properties of the EM FF and thus that they appear to be different, are equivalent. As a consequence, in constructing a unitary and analytic model of the EM structure of any hadron composed of more than two quarks one can employ instead of (4) the simpler set given by (7).

We start with (4). From a direct comparison of the systems (4) and (7) one can see immediately the identity of the first equations.

The second equation in (4), by adding and subtracting the sum $\sum_{j=1}^{n} m_j^4 a_j$, can be modified into the form

$$\sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j^2 a_j - \sum_{j=1}^{n} m_j^4 a_j = 0, \tag{8}$$

from which one can see immediately that the second equation in (7) is fulfilled, as $\sum_{j=1}^{n} m_j^2 a_j = 0$ is just the first equation in (4) and (7) as well.

The third equation in (4), by adding and subtracting the term $\sum_{i=1,i\neq j}^{n} m_i^2 \sum_{j=1}^{n} m_j^4 a_j$ and then subtracting and adding the sum $\sum_{j=1}^{n} m_j^6 a_j$, can be rewritten into the form

$$\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} m_{i_1}^2 m_{i_2}^2 \sum_{j=1}^{n} m_j^2 a_j - \sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j^4 a_j + \sum_{j=1}^{n} m_j^6 a_j = 0, \tag{9}$$

from which, taking into account the first two equations in (7), the third equation of (7) follows.

The fourth equation in (4), adding and subtracting the term $\sum^{n}_{\substack{i_1,i_2=1\\ i_1<i_2, i_r\neq j}} m_{i_1}^2 m_{i_2}^2 \sum_{j=1}^{n} m_j^4 a_j'$, then subtracting and adding the term $\sum_{i=1,i\neq j}^{n} m_i^2 \sum_{j=1}^{n} m_j^6 a_j$ and finally, adding and subtracting the sum $\sum_{j=1}^{n} m_j^8 a_j$, can be transformed into the definitive form

$$\begin{aligned}&\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n} m_{i_1}^2 m_{i_2}^2 m_{i_3}^2 \sum_{j=1}^{n} m_j^2 a_j\\ &- \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} m_{i_1}^2 m_{i_2}^2 \sum_{j=1}^{n} m_j^4 a_j + \sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j^6 a_j - \sum_{j=1}^{n} m_j^8 a_j\\ &= 0,\end{aligned} \tag{10}$$

from which, taking into account the first three equations in (7), the fourth equation in (7) follows.

It is now easy to give a straightforward generalization of the above procedures.

(1) The qth equation in (4) can be decomposed into q-terms (see (8), (9) and (10)) consisting of the product of two parts, where the first part is just the sum of decreasing numbers of products of different vector-meson masses squared, starting from $(q-1)$ coefficients and ending with the constant 1. The second term takes the form $\sum_{j=1}^{n} m_j^{\alpha} a_j$ with an increasing even power α starting from $\alpha = 2$ up to $2q$.

(2) There is alternating sign in front of every term in that decomposition, while the first term is always positive.

Now, in order to carry out a general proof of the equivalence of the two systems of algebraic equations under consideration, let us assume the equivalence of $(m-2)$ equations in (4) and (7). Then, taking into account the generalization of our procedure defined by rules (i) and (ii) above, one can decompose the $(m-1)$th equation in (4) into the following form:

$$\begin{aligned}&\sum_{\substack{i_1,i_2,i_3,\cdots,i_{m-2}=1\\ i_1<i_2<i_3<\cdots<i_{m-2}}}^{n} m_{i_1}^2 m_{i_2}^2 \cdots m_{i_{m-2}}^2 \sum_{j}^{n} m_j^2 a_j\\ &- \sum_{\substack{i_1,i_2,i_3,\cdots,i_{m-3}=1\\ i_1<i_2<i_3<\cdots<i_{m-3}}}^{n} m_{i_1}^2 m_{i_2}^2 \cdots m_{i_{m-3}}^2 \sum_{j=1}^{n} m_j^4 a_j\\ &+ \sum_{\substack{i_1,i_2,i_3,\cdots,i_{m-4}=1\\ i_1<i_2<i_3<\cdots<i_{m-4}}}^{n} m_{i_1}^2 m_{i_2}^2 \cdots m_{i_{m-4}}^2 \sum_{j}^{n} m_j^6 a_j + \cdots\\ &+ (-1)^{m-3} \sum_{i=1}^{n} m_i^2 \sum_{j=1}^{n} m_j^{2(m-2)} a_j\\ &+ (-1)^{m-2} \sum_{j=1}^{n} m_j^{2(m-1)} a_j = 0,\end{aligned} \tag{11}$$

from which one can see immediately that the $(m-1)$th equation in (7) is satisfied as $\sum_{j=1}^{n} m_j^2 a_j = 0$, $\sum_{j=1}^{n} m_j^4 a_j = 0$, $\cdots$, $\sum_{j=1}^{n} m_j^{2(m-2)} a_j = 0$ are just the first $(m-2)$ equations in (7) assumed to be valid.

Finally, we would like to draw attention to the proof of the equivalence of the systems of algebraic equations (4) and (7) from another point of view.

If the sums $\sum_{j=1}^{n} m_j^2 a_j$, $\sum_{j=1}^{n} m_j^4 a_j$, $\sum_{j=1}^{n} m_j^6 a_j$, $\cdots$, $\sum_{j=1}^{n} m_j^{2(m-3)} a_j$, $\sum_{j=1}^{n} m_j^{2(m-2)} a_j$, $\sum_{j=1}^{n} m_j^{2(m-1)} a_j$ are considered to be independent variables, then the first equation in (4) together with the modified forms (8), (9), (10), $\cdots$, (11) form a system of $(m-1)$ homogeneous algebraic equations for these variables and (7) are just its trivial solutions.

4 Conclusions and discussion

Starting from different properties of the EM FF of the hadron we have derived two apparently distinct systems of linear homogeneous algebraic equations for the coupling constant ratios of vector mesons to the hadron under consideration.

For a derivation of the first system of equations we have assumed that the EM FF of the hadron is well approximated by a finite number of vector-meson exchange tree Feynman diagrams leading to the VMD pole parameterization of the FF. The subsequent requirement of true asymptotics of the EM FF gives the system of linear homogeneous algebraic equations for the coupling constant ratios with coefficients that are rather complicated sums of products of squared vector-meson masses.

For a derivation of the second system of equations analytic properties together with the asymptotic behavior of the EM FF have been utilized. The application of the Cauchy theorem to $F_h(t)$, $tF_h(t)$, $t^2F_h(t)$, $\cdots$, $t^{m-2}F_h(t)$ leads to $(m-1)$ integral superconvergent sum rules for $\text{Im}F_h(t)$, $t\text{Im}F_h(t)$, $t^2\text{Im}F_h(t)$, $\cdots$, $t^{m-2}\text{Im}F_h(t)$. Then an appropriate approximation of the imaginary part of the FF by a δ-function gives another system of linear homogeneous algebraic equations for the coupling constant ratios with coefficients that are simply even powers of the vector-meson masses.

By using a sequence of algebraic manipulations it has been proved step by step that both systems of equations for the coupling constant ratios are equivalent.

Finally, the natural question arises of the practical application of such systems of equations for the coupling constant ratios. The latter was already indicated to some extent in the introduction, but in relation to this issue some peculiarities have to be mentioned. The asymptotic behavior of the EM FF is given by the number n_q of constituent quarks in the hadron and so, the system of algebraic equations for coupling constant ratios can be derived only in the case if $n_q > 2$. However, a necessary condition for the latter is saturation of the sum in (1) by the number of vector-meson resonances n being greater than or equal to $n_q - 1$ [9].

If $n > n_q - 1$, then the derived system of equations leads in the construction of a unitary and analytic model of the EM structure of the hadron to a remarkable reduction of the number of free coupling constant ratios.

If $n = n_q - 1$, then adding to $(n-1)$ algebraic equations the equation following from a normalization of $F_h(t)$ at $t = 0$ that typically has a non-zero value, one obtains an inhomogeneous system of n linear algebraic equations with n variables that can be non-trivially solved for. Then solutions are just the numerical values of the coupling constant ratios which appear to be in very good approximation to physical reality (for the case of nucleons see [10]).

The inhomogeneous system of $n \geq n_q - 1$ linear algebraic equations is a very natural tool for a successful simultaneous incorporation of the normalization and the true asymptotics of $F_h(t)$ for the unitary and analytic model of the EM structure of any hadron with $n_q > 2$.

Acknowledgements. This work was in part supported by Slovak Grant Agency for Sciences, Grant No. 2/1111/21 (S.D.) and Grant No. 1/7068/21 (A.Z.D)

References

1. J.J. Sakurai, Ann. of Phys. **11**, 1 (1960)
2. M. Gell-Mann, F. Zachariasen, Phys. Rev. **124**, 953 (1961)
3. V.A. Matveev, R.M. Muradyan, A.N. Tavkhelidze, Lett. Nuovo Cim. **7**, 719 (1973)
4. S.J. Brodsky, G.R. Farrar, Phys. Rev. Lett. **31**, 1153 (1973)
5. S. Dubnička, A.Z. Dubničkova, P. Striženec, Nuovo Cimento A **106**, 1253 (1993)
6. S. Dubnička, A.Z. Dubničkova, J. Kraskiewicz, R. Razcka, Z. Phys. C **60**, 153 (1995)
7. S. Dubnička, Acta Physica Polonica B **27**, 2525 (1996)
8. S. Dubnička, A.Z. Dubničkova, P. Striženec, hep-ph/0108053, and in Proceedings of the International Conference Hadron Structure 2000, Stará lesná 2000 edited by A.Z. Dubničková, S. Dubnička, P. Striženec (Comenius University, Bratislava 2001), p. 204
9. S. Dubnička, A.Z. Dubničkova, P. Weisenpacher, hep-ph/0207223
10. S. Dubnička, A.Z. Dubničkova, E. Krnáč, Phys. Lett. B **261**, 127 (1991)